PMM U.S.S.R.,Vol.54,No.5,pp.663-668,1990
Printed in Great Britian

0021-8928/90 $10.00+0.00

LONGITUDINAL WAVES IN AN ELASTIC MEDIUM WITH A PIECEWISE-LINEAR DEPENDENCE OF THE STRESS ON THE STRAIN*

A.G. KULIKOVSKII and L.A. PEKUROVSKAYA

Longitudinal waves in an elastic body in which the derivative of the stress with respect to the strain (the coefficient of elasticity) can undergo a discontinuity are considered. Fronts can be propagated in such a body, in which a jumplike change in the coefficient of elasticity and the characteristic velocities occurs. It is shown that if the front velocity reaches the value of one of the velocities of the characteristics, reconstruction of the motion occurs, where the special case of the problem of the decay of a weak discontinuity of second-order arises, whereupon a change in the front velocity and type occurs. The problem under consideration is very similar to the problem of the behaviour of an elastic-plastic medium with hardening (although mathematically it does not the same as it), for which the front classification and certain solutions with front reconstruction are given in /1-3/. For a medium of different-modulus, a classification of the front is made in /4, 5/. Some of these fronts are identical with fronts in elastic-plastic media. Solutions of problems of the behaviour of weak first-order discontinuities in a different-modulus elastic body were examined in /4-7/. The paper by G.Ya. Galin, "Phase Transformation Waves" presented at the international conference "Modern Mathematical Problems of Mechanics and Their Application" (Moscow, 1987) was also devoted to flows with a change in properties of the medium.

1. The system of equations describing longitudinal waves in an elastic medium or in a gas in the presence of external mass forces that produce an acceleration $Q(x, t)$ has the form

$$v^{\cdot} - \partial\sigma/\partial x = Q(x, t),\ u^{\cdot} - \partial v/dx = 0,\ S^{\cdot} = 0 \tag{1.1}$$
$$u = u(\sigma, S) = \partial w/\partial x,\ v = w^{\cdot}$$

The dot denotes the partial derivative with respect to time t, x is the Lagrange coordinate, chosen so that the initial density is constant, $w(x, t)$ is the displacement of the medium along the x axis, σ is the stress referred to the initial density, and S is the entropy. The function $u(\sigma, S)$ gives the properties of the medium and is considered known.

We shall assume that the derivative $(\partial u/\partial\sigma)_S$ suffers a discontinuity for $\sigma = \sigma_*(S)$. The first two equations of system (1.1) can be converted to the form

$$v_1^{\cdot} - \partial r/\partial x = 0,\ r = \sigma - \sigma_*,\ v_1 = v - \Phi(x, t) \tag{1.2}$$
$$\lambda^{-2}r^{\cdot} - \partial v_1/\partial x = F(x, t),\ F = \partial\Phi/\partial x,\ \Phi^{\cdot} = Q(x, t) + \partial\sigma_*/\partial x$$

$$\lambda^2 = (\partial r/\partial u)_S$$

when the equality $S = S(x)$ is taken into account. We shall later assume that

$$\lambda^2(r) = \begin{cases} c_*^2 = \text{const}, & r > 0 \\ c^2 = \text{const}, & r < 0 \end{cases} \quad (0 < c_* < c) \tag{1.3}$$

The function Φ is obviously defined apart from an arbitrary function of x whose selection affects the initial conditions for v_1. We shall later use system (1.2) without the subscript 1 on the v.

Introducing the new variable

*Prikl.Matem.Mekhan.,54,5,807-813,1990

$$h(r) = \int_0^r \frac{d\xi}{\lambda(\xi)} = \begin{cases} r/c_*, & h > 0 \\ r/c, & h < 0 \end{cases}$$

we write system (1.2) and its solution in the form

$$\begin{gathered} p^{\cdot} + \lambda \partial p/\partial x = \lambda F, \quad q^{\cdot} - \lambda \partial q/\partial x = \lambda F \\ h = {}^1/_2\,(p + q), \quad v = {}^1/_2\,(q - p) \\ p = \int_{t_0}^{t} \lambda F\, d\tau + f, \quad q = \int_{t_0}^{t} \lambda F\, d\tau + g \end{gathered} \tag{1.4}$$

The integration in the last equalities is performed with respect to time along the appropriate characteristics while f and g are the "initial" values of p and q on these characteristics.

Let us consider the motion of the front separating the domains in which $h > 0,\ \lambda = c_*$ and $h < 0,\ \lambda = c$. We will consider the positive x direction to be the direction from $h > 0$ to $h < 0$.

These are the following possibilities for the velocity of the front W

$$\begin{gathered} A)\ W > c,\ B)\ c_* \leqslant W \leqslant c,\ C)\ -c_* < W < c_*,\ D)\ -c \leqslant W \leqslant -c_*, \\ E)\ W < -c \end{gathered} \tag{1.5}$$

The front D on which h and v are broken is a shock, and the inequality D is its evolutionary condition. The relationships /8, 9/ expressing the continuity of the flux of momentum and displacement of the medium

$$W[v] + [ch] = 0, \quad W[h/c] + [v] = 0 \tag{1.6}$$

should be satisfied on the shock. Continuity of the energy flux in case (1.3) is isolated and serves to calculate the entropy change. The condition of non-decrease of the entropy is satisfied when the inequality D in (1.5) is satisfied /5/. the fronts A, E (fast) and C (slow) are fronts with a continuous change in the quantities that intersect both families of characteristics and were studied earlier /4, 6, 7/. The relationships on them are the condition for the continuity of v and h and the condition of passage $h = 0$. All the quantities are also continuous on the front B but the characteristics with positive velocities take off to both sides from this front. Such fronts were called radiating fronts in /5/. Still another additional relationship should be given in addition to the conditions of continuity and the condition of passage $h = 0$ on the radiating front for its evolution. It can be assumed /5/ that the radiating front itself is a characteristic and the additional relationship thereon is a relationship on the characteristic.

We note that discontinuities, besides those enumerated, that propagate at the characteristic velocities can exist in the domains $h > 0$ and $h < 0$, which can coincide (merge) under certain conditions with one of the continuous fronts listed above. But the discontinuity obtained in this manner is not evolutionary and an infinitesimal variation in the quantities will result in its splitting into two.

2. We will consider singularities in the behaviour of a radiating front. As already mentioned, the front can be regarded as a characteristic (or a narrow bundle of characteristics), and consequently we have

$$p = p_0 + \int F dx, \quad p_0 = \text{const}, \quad q = g\,(x + ct) + \int F dx \tag{2.1}$$

in the domain ahead of the front occupied by the characteristics departing from the front.

As in (1.4), integration in the expression for q is performed along the characteristics $dx/dt = -c$ in the xt plane, and in the expression for p, first along a segment of the radiating front trajectory (KL in Fig.1), and then along the characteristic $dx/dt = c$ (LM in Fig.1). The passage condition $p + q = 0$ should be satisfied on all lines of the front KLN. Using this equality at the points L and N we evaluate h at the points M and M'

$$\begin{gathered} 2h_M = p_M + q_M = \int_{LM} F\, dx - \int_{LN} F\, dx + \int_{MN} F\, dx = -\iint_{\Delta LMN} F^{\cdot}\, dx\, dt \\ 2h_{M'} = p_{M'} + q_{M'} = \int_{LM'} F\, dx - \int_{LN} F\, dx - \int_{NM'} F\, dx = \iint_{\Delta LM'N} F^{\cdot}\, dx\, dt \end{gathered} \tag{2.2}$$

It is seen that the first partial derivatives of h are equal to zero on both sides of the radiating front and the second derivatives along the normal to the line of the front, at points where $F^{\cdot} \neq 0$, separate and have different signs on different sides of the front. Since there should be $h_M \geqslant 0$ and $h_{M'} \leqslant 0$, it is necessary that

$$F^{\cdot} \leqslant 0 \tag{2.3}$$

Another necessary condition for the existence of a continuous radiating front is that the equation of its motion $x = x_*(t)$, determined by the condition $p + q = 0$, where p and q are given by the equalities (2.1), should satisfy the evolutonarity conditions (inequality B in (1.5), where $W = dx_*/dt$). The quantity W is found as a result of double differentiation of the left-hand side of the equality $p + q = 0$ along the front

$$F + g' = 0 \tag{2.4}$$

$$(F^{\cdot} + cg'')\,dt + (\partial F/\partial x + g'')\,dx_* = 0, \quad \frac{dx_*}{dt} \equiv W = -\frac{F^{\cdot} + cg''}{\partial F/\partial x + g''}$$

Breakdown of the inequality (2.3) and the evolutionarity conditions can result in replacement of the radiating front by a front of another kind. Thus, if W coincides with c_* or c, then it can be expected that the radiating front (B) is transformed into the slow (C) or fast (A) continuous front. Such a transformation will be considered below as the parameters of the problem change.

Let us clarify what will occur with the radiating front when condition (2.3) is violated. Let the dashed line in Fig.1 be given by the equation $F^{\cdot} = 0$, where $F^{\cdot} < 0$ below it and $F^{\cdot} > 0$ above. Without changing the area of the triangle LMN and shifting it along the front it is obviously possible to place the point M such that the integral of $F^{\cdot}$ over the area of the triangle will vanish and, therefore, $h = 0$ at the point M. This means that a certain line GM on which $h = 0$ stands off from the point G on the radiating front where $F^{\cdot} = 0$. Depending on its slope at the point G (which depends, in turn, on the slope of the line $F^{\cdot} = 0$) the line mentioned can be either a non-radiating fast front (A) or a radiating front (B). In both cases the front velocity at the point G changes jumpwise. The presence of this new front GM obviously changes the solution for $t > t_G$ and the line GN will no longer be a radiating front.

The front behaviour presented above is associated with the fact that the expansion of $h(x, t)$ in the neighbourhood of a point on the radiating front starts with second-order terms for $F^{\cdot} \neq 0$, and the expansion at the point G starts with third-order terms for $F^{\cdot} = 0$. For this reason, (2.5) for W that does not contain the third derivative becomes invalid at the point G.

3. We will consider behaviour of the fronts separating the domains where the characteristic velocities take the values c and c_*. The types of these fronts that satisfy the evolutionarity condition were indicated in (1.3). By virtue of the evolutionarity a small change in the perturbations arriving at the front results in a small change in the velocity of the front and of the departing perturbations. Non-trivial behaviour of the solution can be expected on the boundaries of front existence. One example of such behaviour is examined in Sect.2.

The reconstruction of the solution when the velocity of the fast front (A) reaches the value c at a certain point x, t will be studied below. We will select this point as the origin and we will seek the solution in the form of series in x and t. Retaining terms not higher than the first degree in the expansion of F and taking into account that the function F is determined apart from an arbitrary function of x, as was noted in Sect.1, we set $F = bt$. Moreover, without loss of generality, it can be assumed that $p = q = 0$ at the origin.

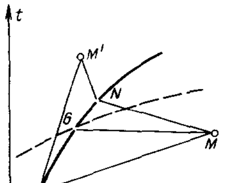

Fig.1

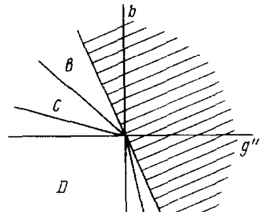

Fig.2

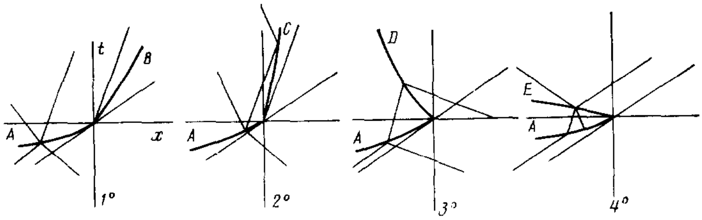

Fig.3

Expanding expression (1.4) for the invariants of p and q arriving at the discontinuity in series and limiting ourselves to terms not higher than quadratic, we obtain

$$p = f'(x - ct) + {}^1/_2 f''(x - ct)^2 + {}^1/_2 bct^2 \qquad (3.1)$$
$$q = g'(x + ct) + {}^1/_2 g''(x + ct)^2 + {}^1/_2 bct^2$$

Here f', f'', g' and g'' are derivatives of functions that yield the initial conditions for p and q, with respect to the arguments of these functions taken for $x = 0, t = 0$. Considering the small distance $\Delta t = t - x/c$ in the direction t between the front and the characteristic touching the front at the origin and retaining only terms linear in Δt, we obtain, using (3.1), the equation of the front (A)

$$2h = p + q = -f'c(t - x/c) + 2g'x + (2g'' + b/c)x^2 = 0 \qquad (3.2)$$

Using the notation $dx/dt = W$, we obtain from the last equality that the condition $W = c$ for $t = 0$, $W > c$ for $t < 0$ and the condition $h^{\cdot} > 0$ on the front can be written in the form .

$$g' = 0, \ 2g'' + b/c < 0, \ f' < 0 \qquad (3.3)$$

As will be shown below, continuation of the solution in t depends on the values of b and g''. The second equality of (3.3) prohibits the point with coordinates g'', b from lying in the shaded part of the plane in Fig.2.

Let us examine possible cases of front continuation.

1°. Continuation of the interfacial boundary, the radiating front B (Fig.3, 1°) on which according to (2.4)

$$F + \partial q/\partial x = bt + g''(x + ct) = 0, \ b/g'' = \alpha_B(W) \equiv -W - c$$

the evolutionarity condition $c_* \leqslant W \leqslant c$ yields

$$-2c \leqslant b/g'' \leqslant -(c + c_*) \qquad (3.4)$$

(the domain B in Fig.2).

2°. Continuation of the interfacial boundary, the slow front C (Fig.3, 2°) and to which the characteristics arrive from both the domain ahead of the front and from the domain behind the front. Using the continuity of p on the front A and expression (3.1) for p on the arriving characteristics, we find an expression for p in the domain where $\lambda = c_*, h > 0$ (later the quantities referring to this domain will be given asterisks while the quantities without the asterisks will refer to the domain where $\lambda = c$, $h < 0$)

$$p_* = P(x - c_*t)^2 + {}^1/_2 bc_*t^2, \quad P = -[2c^2g'' + {}^1/_2 b(c + c_*)](c - c_*)^{-2} \qquad (3.5)$$

The equation of the front is given by the equation $p_* + q = 0$, and the quantity q is defined by (3.1). Dividing the last equality by g'' we find

$$b/g'' = \alpha_C(W) = -[(3c - c_*)W - (3c_* - c)c](W - 2c_* + c)^{-1} \qquad (3.6)$$

The function $\alpha_C(W)$ changes monotonically in the interval $-c_* < W < c$ where the slow front is evolutionary and takes on the following values at the ends of this interval:

$$\alpha_C(c_*) = -(c + c_*), \ \alpha_C(-c_*) = -[(c - c_*)^2 - 4cc_*](c - 3c_*)^{-1}$$

They bound a domain of values (sector) in the g'', b plane for which the solution contains the slow front. One of the boundaries of this sector $b = \alpha_C(c_*)g''$ coincides with the boundaries of the domain of existence of the radiating front B where the front velocity changes continuously when this boundary is crossed. Depending on the magnitude of the ratio c/c_* another boundary $b = \alpha_C(-c_*) g''$ can lie in the second, third, or fourth quadrants (Fig.2).

3°. Continuation of the interfacial boundary, the shock D (Fig.3, 3°). Substituting (1.4) for v and h on both sides of the discontinuity in terms of p, q (from the right of the discontinuity) and p_*, q_* (from the left of the discontinuity) into (1.6) and eliminating p from these relationships, we obtain

$$W = c_* [(c + c_*) q_* - (c - c_*) p_* - 2cq] [2c_* q - (c + c_*) q_* - (c - c_*) p_*]^{-1} \tag{3.7}$$

The shock-wave velocity is expressed here in terms of the values of the invariants corresponding to characteristics arriving at the wave, on the shock.

Using the continuity of the invariant q for passage through the front A, we find

$$q_* = [2c^2 g'' + {}^1/_2 b (c - c_*)](c + c_*)^{-2}(x + c_* t)^2 + {}^1/_2 b c_* t^2 \tag{3.8}$$

Substituting (3.5) and (3.8) into (3.7) in which we set $x = Wt$, we obtain

$$b/g'' = \alpha_D(W) \equiv [(c_*^2 - 5c^2) W - (c^2 - 5c_*^2) c][2cW - 3c_*^2 + c^2]^{-1} \tag{3.9}$$

The function $\alpha_D(W)$ varies monotonically in the shock interval of evolutionarity $-c \leqslant W \leqslant -c_*$ and takes the following values at its ends:

$$\alpha_D(-c_*) = [(c - c_*)^2 - 4cc_*](3c_* - c)^{-1}, \quad \alpha_D(-c) = -4c(c^2 + c_*^2)(c^2 + 3c_*^2)^{-1}$$

Hence, it is seen that one of the boundaries $b = \alpha_D(-c_*) g''$ of the domain where the solution contains a shock wave coincides, in the plane g'', b, with the boundary obtain above for the domain where the solution contains the slow front C, where W changes continuously during passage through this boundary. The other boundary is given by the equality $b = \alpha_D(-c) g''$ and lies in the fourth quadrant.

4°. Continuation of the interfacial boundary, the fast front E after whose passage the characteristic velocity of the medium changes from the value c_* to the value c (Fig.3, 4°). The equation of the front is $p_* + q_* = 0$. Substituting (3.5) and (3.8) into this equation we find

$$b/g'' = \alpha_E(W) = 8c^2 (c_*^2 - cW)[(3c^2 + c_*^2) W - c(5c_*^2 - c^2)]^{-1} \tag{3.10}$$

Since W can become infinite, it is convenient to write the domain of evolutionarity of discontinuities of this type in the form

$$-c^{-1} < W^{-1} < c^{-1}$$

As W^{-1} varies within this interval, the function $\alpha_E(W)$ changes monotonically and takes the following values at its ends:

$$\alpha_E(-c) = -4c(c^2 + c_*^2)(c^2 + 3c_*^2)^{-1}, \quad \alpha_E(c) = -2c$$

In the g'', b plane, the boundaries of the domain for whose points a front of the type under consideration is realized in the solution are given by the equations $b = \alpha_E(-c) g''$ and $b = \alpha_E(c) g''$. The first of these boundaries coincides with the boundary of the domain of the shock wave D, where W changes continuously during passage through this boundary, while the second coincides with the boundary of the domain of the existence of the front A (see the second inequality in (3.3)).

The investigation presented shows that if the velocity of the fast front reaches the boundary-value $W = c$, then in the case of the common position when the expansions (3.1) satisfying conditions (3.3) are valid for the arriving perturbations, continuation of the solution can always be constructed. It contains a front whose type and velocity are determined uniquely by the values g'' and b in (3.1) (the values of f' and f'' in the approximation under consideration do not affect the solution). The domain for whose points there are no other solutions corresponds to the solutions containing the radiating front in the g'', b plane.

Problems concerning reconstructing solutions for achieving evolutionarity boundaries by the velocity of fronts of other types can be solved analogously.

We note that if $F \equiv 0$, as it was in /4-6/, then the invariants will be conserved along the characteristics and the radiating front will never occur (an entire domain can only occur in the xt plane for constant values of all the quantities). As has been shown, since it is possible to set $F = bt, b = \text{const}$ in the neighbourhood of each point, and the influence of F is felt in the neighbourhood of this point in the form of quadratic terms in t, it is conceivable that the need to introduce radiating fronts would also not appear in the solution of problems of the decay of an arbitrary weak first-order discontinuity.

REFERENCES

1. LEE E.N., A boundary-value problem in the theory of plastic wave propagation, Quart. Appl. Math., 10, 1953.
2. CLIFTON R.J. and TING T.C.T., The elastic-plastic boundary in one-dimensional wave propagation. J. Appl. Mech., 35, 2, 1968.
3. TING T.C.T., On the initial speed of elastic-plastic boundaries in longitudinal wave propagation in a rod, J. Appl. Mech., 38, 1971.
4. MASLOV V.P., MOSOLOV P.P. and SOSNINA E.V., On types of discontinuities of solutions of the equations of longitudinal, free, one-dimensional motion in an elastic medium of different modulus in: Problems of Non-linear Mechanics of a Continuous Medium, Valguz, Tallinn, 1985.
5. KULIKOVSKII A.G. and PEKUROVSKAYA L.A., On the fronts of strong and weak discontinuities in solutions of the equations of different-modulus elasticity theory, PMM, 53, 2, 1989.
6. MASLOV V.P. and MOSOLOV P.P., General theory of the solutions of the equations of motion of a different-modulus elastic medium, PMM, 49, 3, 1985.
7. MASLOV V.P., and MOSOLOV P.P., Elasticity Theory for a Different-modulus Medium, Izd. MIEM, Moscow, 1985.
8. BLAND D., Non-linear Dynamical Theory of Elasticity, Mir, Moscow, 1972.
9. LAX P.D., Hyperbolic system of conservation laws, Comm. Pure and Appl. Math., 10, 4, 1957.

Translated by M.D.F.

PMM U.S.S.R., Vol.54, No.5, pp.668-677, 1990
Printed in Great Britain

0021-8928/90 $10.00+0.00

MATERIAL AND SPATIAL REPRESENTATIONS OF THE CONSTITUTIVE RELATIONS OF DEFORMABLE MEDIA*

G.L. BROVKO

The problem of giving a basis to material and spatial representations for the constitutive relations (CR) of media, their correspondence (equivalence) to each other, as well as the problem of the explicit resolution of implicit forms of CR (in material and spatial representations) are examined from the point of view of the general theory of constitutive relationships of the classical mechanics of a continuous medium, based on the principles of determinism and causality, locality, independence of the reference system, and the hypothesis of macrophysical determinacy.

Approaches based on the introduction of spatial-type tensors, used in an Euler description (/20-28/, say) are used in addition to the traditional approaches of the mechanics of a continuous medium that are in direct agreement with the macroscopic determinacy hypothesis /1, 2/, and expressed from the beginning, as a rule, in the terminology of the material-type tensors utilized in the Lagrange description of motion of a medium (see /2-7/, say), or explicitly understood by the connection with such tensors (for instance /8-19/). Numerous papers on plasticity that propose extrapolation of the CR, known for small deformations, by some method to the case of finite deformations in an Euler description are among them.

Important questions arise here, in principle: 1) is such extrapolation legitimate from the point of view of the general classical theory of CR, i.e., does the CR obtained agree, in principle, with the postulate of macroscopic determinability? (the example in /28/ is one of the modifications of such an erroneous inconsistency), 2) which is the spatial representation (Euler form) of the constructed or known material (Lagrange) CR and vice-versa? 3) if the CR is constructed in implicit form, especially in the spatial tensor

**Prikl.Matem.Mekhan.*, 54,5,814-824,1990